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On the Tropicalization of the Hilbert Scheme

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Abstract

In this article we study the tropicalization of the Hilbert scheme and its suitability as a parameter space for tropical varieties. We prove that the points of the tropicalization of the Hilbert scheme have a tropical variety naturally associated to them. To prove this, we find a bound on the degree of the elements of a tropical basis of an ideal in terms of its Hilbert polynomial.
As corollary, we prove that the set of tropical varieties defined over an algebraically closed valued field only depends on the characteristic pair of the field and the image group of the valuation.
In conclusion, we examine some simple examples that suggest that the definition of tropical variety should include more structure than what is usually considered.

1. Introduction

In [17] Speyer and Sturmfels studied the tropicalization of the Grassmannian and found that it is a parameter space for tropical linear subspaces, just like the ordinary Grassmannian in algebraic geometry. This result inspired our study of the tropicalization of the Hilbert scheme for two reasons: first, Grassmannians are an example of Hilbert

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schemes, and second, the standard construction of the Hilbert scheme realizes it as a subscheme of a Grassmannian.

The Hilbert scheme is a parameter space for embedded projective varieties. For this reason, it is natural to ask whether its tropicalization can be considered as a parameter space for tropical varieties: the idea is to construct a sort of tropical Hilbert scheme. In this paper we prove the following result in this direction:

Theorem 4.2: *Let $\mathbb{K}$ be an algebraically closed field with surjective real-valued valuation. There is a commutative diagram*

$$\begin{array}{ccc} \mathrm{Hilb}_n(p) & \stackrel{b}{\longleftrightarrow} & \{V \subset \mathbb{KP}^n \mid V \text{ has Hilbert polynomial } p\} \\ \downarrow \tau & & \downarrow \mathrm{Trop} \\ \mathrm{Trop}(\mathrm{Hilb}_n(p)) & \stackrel{s}{\twoheadrightarrow} & \{\mathrm{Trop}(V) \subset \mathbb{TP}^n \mid V \text{ has Hilbert polynomial } p\} \end{array}$$

Where b is the classical correspondence between points of the Hilbert scheme and subschemes of $\mathbb{KP}^n$. The map s is surjective.

In order to prove this theorem, we need some properties of tropical bases. We recall that a tropical basis for an ideal I is a finite set of polynomials generating the ideal and such that the intersection of the hypersurfaces defined by these polynomials is equal to the tropicalization of the variety defined by I. Tropical bases are a fundamental tool in computational tropical geometry, but not very much is known about them. Here we find the following property, that is of independent interest, and it is the main technical result of the paper:

Theorem 3.7: *Let I be a saturated homogeneous ideal, with Hilbert polynomial p. Then there exist a tropical basis consisting of polynomials of degree not greater than the Gotzmann number of p. In particular this bound only depends on p.*

This result is a sort of converse of the result proved in [12], where, given an ideal, they find a tropical basis consisting of a small number of polynomials, which can have high degree.

The proof of this theorem is based on the following technical lemma, whose proof uses primary decompositions of polynomial ideals and some commutative algebra.

Lemma 5.5: *Let I be a saturated ideal of polynomials. If I contains a monomial, then it contains a monomial of degree not greater than its arithmetic degree.*

To relate this with the Hilbert polynomial, we prove the following result, whose proof draws heavily from [11]:

Theorem 5.1: *Let I be a homogeneous ideal with Hilbert polynomial p. Then the arithmetic degree of I is not greater than the Gotzmann number of p. Moreover the bound is optimal: For every Hilbert polynomial p, there exists an ideal I with Hilbert polynomial p and such that the arithmetic degree of I is exactly the Gotzmann number of p.*

We also point out a nice corollary of theorem 4.2. The definition of tropical variety depends on the choice of a valued field $\mathbb{K}$, that is used for their construction. For an example of this, see [17, Thm. 7.2], where there is an example of dependence on the characteristic of $\mathbb{K}$. One can construct larger and larger valued fields, and it is *a priori* possible to think that if the field is sufficiently large, new tropical varieties may appear. As a corollary of theorem 4.2, we are able to say that this never happens. More precisely:

Theorem 4.4: *The set of tropical varieties definable over an algebraically closed valued field only depends on the characteristic pair of the valued field and on the image of the valuation. If one considers valuations that are surjective on $\mathbb{R}$, the set of tropical varieties only depends on the characteristic pair.*

This statement, which is of independent interest, has, *a priori*, nothing to do with the Hilbert scheme, but at the best of our knowledge, no other proof of this result exists in literature.

In the final part of the paper, we come back to the idea of using the tropicalization of the Hilbert scheme as a parameter space for tropical varieties. As we said, the application that associates a tropical variety to a point of the tropicalization of the Hilbert scheme is surjective (on the set of all tropical varieties that are tropicalizations of algebraic varieties with a fixed Hilbert polynomial) but is not in general injective. To understand why this happens we studied two kinds of examples. The first is about the Hilbert schemes of hypersurfaces. Here one problem is that there are several different tropical polynomials that define the same function, and thus the same hypersurface, whenever one of the terms never achieves the maximum. However, it is possible to adjust things so that the map becomes injective. It is necessary to add some extra structure to the tropical hypersurfaces, namely to add weights to the maximal faces, as usual. Once this extra structure is considered, there exists a unique subpolyhedron $P \subset \operatorname{Trop}(\operatorname{Hilb}_n(p))$ such that the restriction of the correspondence to P is bijective.

We think that this property of the existence of a subpolyhedron that is a "good" parameter space should probably be true also for the general Hilbert scheme, but in general it is not clear what is the suitable extra structure. The second kind of examples is about the Hilbert schemes of the pairs of points in the tropical plane. In this case adding the weights to the tropical variety structure is not enough and this is because the tropicalization of the Hilbert scheme seems to remember more information about the non reduced structure than is expressible by a single integer. This suggests that it would be necessary to enrich the structure of tropical variety even more.

For the convenience of the reader we include here a brief overview of each section. Section 2 opens with some essential facts on the Hilbert polynomial and the Gotzmann number. It then proceeds describing the natural embedding of the Hilbert scheme in the projective space, through the Grassmannian. This is the embedding that will be used for the tropicalization. Section 3 contains some general facts on valued fields, which are necessary to prove theorem 4.4, along with some fundamental facts about tropical varieties. At the end of the section we state and prove theorem 3.7, although the proof of a technical lemma is postponed to section 5. The main theorem, 4.2, and its corollary, theorem 4.4, on the dependence on the base field are stated and proved in section 4. Section 5 itself is completely devoted to proving lemma 5.5 which is a key part of the proof of theorem 3.7. To do so we use primary decomposition to investigate the degree of monomials contained in an ideal, then we use the arithmetic degree, see theorem 5.1, to relate the estimates with the Gotzmann number. In the last section we treat our two examples: hypersurfaces and pairs of points in the projective plane.

2. Preliminaries

2.1. The Hilbert Polynomial

Let k be a field and $S = k[x_0, \ldots, x_n]$, the free graded k-algebra of polynomials in $n+1$ variables. We denote by $S_d \subset S$ the vector subspace of homogeneous polynomials of degree d. If I is a homogeneous ideal, we denote its homogeneous parts by $I_d = I \cap S_d$. The quotient algebra has a natural grading $S/I = \bigoplus_d (S/I)_d = \bigoplus_d S_d/I_d$. The **Hilbert function** of I is the function $h_I : \mathbb{N} \longrightarrow \mathbb{N}$ defined by:

$$h_I(d) = \dim_k(S_d/I_d) = \dim_k S_d - \dim_k I_d$$

It is not difficult to see ([8, Thm. 1.11]) that for every homogeneous ideal I there exists a number d_0 and a polynomial p_I of degree $s \leq n$, with rational coefficients, such that for $d \geq d_0$ we have

$$h_I(d) = p_I(d)$$

The polynomial p_I is the **Hilbert polynomial** of I. Note that $p_I = p_{I^{\mathrm{sat}}}$, hence the Hilbert polynomial only depends on the subscheme of $\mathbb{P}^n$ defined by I. Many invariants of this subscheme may be read from p_I, for example the projective dimension of the subscheme, that we denote here by $\dim(I) = s = \deg(p_I)$, and the degree of the subscheme, that we denote here by $\deg(I) = s!a_s$, where $a_s x^s$ is the term of higher degree of p_I.

A **numerical polynomial** is a polynomial with rational coefficients taking integer values for all large enough integer arguments. This includes the Hilbert polynomials of homogeneous ideals. Given a finite sequence of numbers $m_0, \ldots, m_s \in \mathbb{Z}$, with $m_s \neq 0$, the following formula defines an integer polynomial in x of degree s with term of higher degree $\frac{m_s}{s!} x^s$:

$$\mathcal{P}(m_0, \ldots, m_s; x) = \sum_{i=0}^{s} \binom{x+i}{i+1} - \binom{x+i-m_i}{i+1}$$

Moreover, by [2, Lemma 1.3], every numerical polynomial of degree s can be expressed in the form $\mathcal{P}(m_0, \ldots, m_s; x)$.

There is a simple description of which numerical polynomials are the Hilbert polynomial of some homogeneous ideal: Let $m_0, \ldots, m_s \in \mathbb{Z}$ with $m_s \neq 0$. Then there exists a non-zero homogeneous ideal $I \subset S$ with $p_I = \mathcal{P}(m_0, \ldots, m_s; x)$ if and only if $s < n$ and $m_0 \geq \cdots \geq m_s > 0$ (see [11, Cor. 5.7]).

We conclude with some examples. If $I = (0)$, then $p_I(x) = \mathcal{P}(m_0, \ldots, m_n; x)$ with $m_0 = \cdots = m_n = 1$ (see [2, Chap. 2, 1.15]). If $I = (f)$ is a principal ideal with $\deg(f) = d$, then $p_I(x) = \mathcal{P}(m_0, \ldots, m_{n-1}; x)$ with $m_0 = \cdots = m_{n-1} = d$ (see [2, p. 30]). At the other extreme, if $\dim(I) = 0$, with $\deg(I) = d$, then $p_I(x) = \mathcal{P}(d; x) = d$. If I defines a curve of degree d and arithmetic genus g, then $p_I(x) = \mathcal{P}(\binom{d}{2} + 1 - g, d; x)$ (see [2, Chap. 2, 1.17]). The other easy case is when I is generated by linear forms, and $\dim(I) = s$. Then $p_I(x) = \mathcal{P}(m_0, \ldots, m_s; x)$ with $m_0 = \cdots = m_s = 1$.

2.2. Gotzmann number

The expression $\mathcal{P}(m_0, \dots, m_s; x)$ that we used for the Hilbert polynomial is not the only one in use. Notably, there is the expression used in the statements of Gotzmann's regularity and persistence theorems ([10], [6, Sect. 4.3]):

$$p_I(x) = \sum_{i=1}^{r} \binom{x + a_i - i + 1}{a_i}$$

with $a_1 \geq \dots \geq a_r \geq 0$.

Since we will make use of Gotzmann's theorems later on, we find it appropriate to discuss the conversion between this form and the one we introduced earlier. In particular, we are interested in the **Gotzmann number**, which is r in this last expression and m_0 in the earlier one. We thank Diane Maclagan for having suggested to us both the statement and the proof of the next lemma.

Lemma 2.1 Let f be a polynomial, and suppose that $f = \mathcal{P}(m_0, \dots, m_s; x)$ and $f = \sum_{i=1}^{r} \binom{x+a_i-i+1}{a_i}$. Then each m_k is equal to the number of a_is greater or equal to k. In particular, $m_0 = r$.

Proof. The proof consists mostly in reordering the sums and in a repeated use of the following property of binomials:

$$\binom{\alpha}{\beta} + \binom{\alpha}{\beta + 1} = \binom{\alpha + 1}{\beta + 1}. \tag{1}$$

By applying repeatedly the (1) we can change the first form as follows:

$$\sum_{i=0}^{s} \binom{x+i}{i+1} - \binom{x + i - m_i}{i+1} = \sum_{i=0}^{s} \sum_{j=1}^{m_i} \binom{x+i-j}{i}.$$

This form is also unique for any given polynomial because the original one had this property and they depend on the same set of parameters. We will now consider the second form for the Hilbert polynomial and try to modify it into this one.

Now note that the degree of a polynomial is s when expressed in the first form and is equal to a_1 in the second one. This means that $a_i \leq s$ for all i and hence if we go through the integers between 0 and s we will find all the values of the a_is. We can then write

$$\sum_{i=1}^{r} \binom{x + a_i - i + 1}{a_i} = \sum_{i=0}^{s} \sum_{j \mid a_j = i} \binom{x + i - j + 1}{i}$$

Note that in this expression the second sum is more akin of a "selection" but writing it in this form has the advantage of making clearer the fact that it can commute with the other sums. Now we apply the (1) repeatedly:

$$= \sum_{i=0}^{s} \sum_{j \mid a_j = i} \sum_{k=0}^{i} \binom{x + k - j}{k}$$

and reorder the sums:

$$= \sum_{k=0}^{s} \sum_{i=k}^{s} \sum_{j|a_j=i} \binom{x+k-j}{k}$$

All the two inner sums do is to select the js for which $a_j \geq k$. Since the expression inside the sums does not depend on i we can get rid of it. Finally we can make one last transformation based on the fact that the a_is are ordered.

$$= \sum_{k=0}^{s} \sum_{j|a_j \geq k} \binom{x+k-j}{k} = \sum_{k=0}^{s} \sum_{j=1}^{\max_i\{a_i \geq k\}} \binom{x+k-j}{k}.$$

By comparing this last form with the one we obtained at the beginning, we can say that $m_i = max_i\{a_i \geq i\}$ which, since the a_is are ordered decreasingly, is the same as the number of a_is that are greater or equal to i. □

2.3. The embedding of the Hilbert Scheme

The **Grassmannian** $\mathbb{G}(n,d)$ is the set of all the d-dimensional vector subspaces of the vector space k^n. Equivalently, the Grassmannian can be regarded as the set of the $(d-1)$-dimensional projective subspaces of $\mathbb{P}(k^n)$.

The Grassmannian can be embedded naturally in a projective space via the Plücker embedding. Let $L \subseteq k^n$ be a d-dimensional vector subspace and let $v_1, \dots, v_d$ be a basis of L. The embedding can be defined as follows:

$$\begin{array}{ccc} \mathbb{G}(n,d) & \hookrightarrow & \mathbb{P}\left(\bigwedge^d k^n\right) \\ \langle v_1, \dots, v_d \rangle & \longmapsto & [v_1 \wedge \cdots \wedge v_d] \end{array}$$

It is easy to see that, if one considers two different bases for L, the wedge products of the elements of each base will only differ by a multiplicative constant, meaning that this map is well defined in the projective space. Moreover this map is injective, because knowing $v_1 \wedge \cdots \wedge v_d$ you can write equations for $\langle v_1, \dots, v_d \rangle$. This gives an embedding of $\mathbb{G}(n,d)$ into $\mathbb{P}\left(\bigwedge^d k^n\right) = \mathbb{P}\left(k^{\binom{n}{d}}\right)$. The image of this embedding is the set of alternate tensors of rank 1. Equations for this subset are given by the Plücker ideal, an ideal generated by quadratic polynomials with integer coefficients. This identifies the Grassmannian with a projective algebraic variety.

The Grassmannian can be used to construct the embedding of the Hilbert scheme that we will use later for its tropicalization.

Fix a numerical polynomial $p(x) := \mathcal{P}(m_0, \dots, m_s; x)$ and a projective space $\mathbb{P}^n$, such that $s < n$ and $m_0 \geq \cdots \geq m_s > 0$. It is possible to parametrize all closed subschemes of $\mathbb{P}^n$ having Hilbert polynomial p with the closed points of a scheme $\mathrm{Hilb}_n(p)$. This scheme is called the **Hilbert scheme** and is a fine moduli space for the closed subschemes of $\mathbb{P}^n$ up to identity. Now we describe the projective embedding of this scheme that we will use for the tropicalization.

The **Castelnuovo-Mumford regularity** of a saturated homogeneous ideal I, denoted by $\mathrm{reg}(I)$, is the smallest integer m such that I is generated in degree not greater than m and, for every i, the i-th syzygy module of I is generated in degree not greater than $m+i$ (see [8, sec. 20.5 and ex. 20.20], [2, Chap. 2, 2.1 and Lemma 2.4]). For every $d \geq \operatorname{reg}(I)-1$ we have $h_I(d) = p_I(d)$ (see [2, Chap. 2, 2.5]). The fundamental fact is that for every saturated ideal I its Castelnuovo-Mumford regularity is at most equal to the Gotzmann number, which only depends on the Hilbert polynomial of I. This is Gotzmann's regularity theorem (see [10], [6, Sect. 4.3, Thm. 4.3.2], [2, Chap. 2, Prop. 9.4 and Prop. 10.1]).

Consider the Gotzmann number m_0. We have that $I = \langle I_{m_0} \rangle^{\text{sat}}$, hence I is determined by its component of degree m_0. The space S_{m_0} is a vector space of dimension $N := \binom{n+m_0}{n}$ and I_{m_0} is a vector subspace of codimension $p(m_0)$. We can define the injective map

$$\begin{array}{ccc} \{I \subset S \mid I \text{ saturated ideal}, p_I = p\} & \hookrightarrow & \mathbb{G}(N, N-p(m_0)) \\ I & \mapsto & I_{m_0} \end{array}$$

Now we want to describe the equations of this embedding of the Hilbert scheme, at least from a set-theoretical viewpoint. To do this we need the following fact. Given $s < n$ and $m_0 \geq \cdots \geq m_s > 0$, let $I \subset S$ be any homogeneous ideal (possibly non-saturated, with any Hilbert polynomial), and choose $m \geq m_0$. If $h_I(x) \leq \mathcal{P}(m_0, \ldots, m_s; x)$ for $x = m$, then this holds for every $x \geq m$. If $h_I(x) = \mathcal{P}(m_0, \ldots, m_s; x)$ for $x = m, m+1$, then this holds for every $x \geq m$. This result comes directly from Gotzmann's persistence theorem (see [2, Chap. 2, Prop. 9.5 and Prop. 10.2], [6, Sect. 4.3, Thm 4.3.3]).

This fact gives the equations: we have to consider only the ideals $I \subset S$ such that $\dim(I_{m_0}) = N - p(m_0)$ and $\dim(I_{m_0+1}) \leq \binom{n+m_0}{n-1} - p(m_0+1)$. Expressed in the Plücker coordinates, these relations give equations for the image that are polynomials with integer coefficients, see [2, Chap. 6, 1.2]. Hence we have identified the Hilbert scheme $\operatorname{Hilb}_n(p)$ with a projective algebraic subvariety of the Grassmannian $\mathbb{G}(N, N-p(m_0))$.

Composing this embedding with the Plücker embedding allows us to embed the Hilbert scheme in a projective space as follows: given a homogeneous saturated ideal I with Hilbert polynomial p let $v_1, \ldots, v_d$ be a basis of I_{m_0} as a vector space. Then the point of $\operatorname{Hilb}_n(p)$ corresponding to I is sent into $[v_1 \wedge \cdots \wedge v_d]$.

3. Tropical Varieties

3.1. Valued fields

Let $\mathbb{K}$ be an algebraically closed field, with a real-valued valuation $v : \mathbb{K}^* \mapsto \mathbb{R}$. We will always suppose that 1 is in the image of the valuation.

As $\mathbb{K}$ is algebraically closed, the image of v is a divisible subgroup of $\mathbb{R}$, that we will denote by Λ_v. This subgroup contains $\mathbb{Q}$, hence it is dense in $\mathbb{R}$. Moreover, the multiplicative group $\mathbb{K}^*$ is a divisible group, hence the homomorphism $v : \mathbb{K}^* \mapsto \Lambda_v$ has a section, i.e. there exists a group homomorphism $t^\bullet : \Lambda_v \longrightarrow \mathbb{K}^*$ such that $v(t^w) = w$ (see [14, lemma 2.1.13]). Note that $t^\bullet$ satisfies $t^{w+w'} = t^w t^{w'}$. We will denote t^1 by $t \in \mathbb{K}$.

We denote by $\mathcal{O} = \{x \in \mathbb{K} \mid -v(x) \leq 0\}$ the **valuation ring** of $\mathbb{K}$, and by $\mathfrak{m} = \{x \in \mathbb{K} \mid -v(x) < 0\}$ the unique maximal ideal of $\mathcal{O}$. The quotient $\Delta = \mathcal{O}/\mathfrak{m}$ is the **residue field** of $\mathbb{K}$, an algebraically closed field. The **characteristic pair** of the valued field $\mathbb{K}$ is the pair of numbers $\mathrm{cp}(\mathbb{K}, v) = (\mathrm{char}(\mathbb{K}), \mathrm{char}(\Delta))$. This pair can assume the values $(0,0)$, $(0,p)$ or (p,p), where p is a prime number.

In the following we will need to consider valued fields such that the image of the valuation is surjective into $\mathbb{R}$. Examples of such fields are provided by the fields of **transfinite Puiseux series**, also called the field of generalized power series, or the Malcev-Neumann ring. If Δ is an algebraically closed field, the set

$$\Delta((t^{\mathbb{R}})) = \left\{ \sum_{w \in \mathbb{R}} a_w t^w \mid \{w \in \mathbb{R} \mid a_w \neq 0\} \text{ is well ordered } \right\}$$

endowed with the operations of sum and product of formal power series is an algebraically closed field, with surjective real-valued valuation $v(\sum_{w\in\mathbb{R}} a_w t^w) = \min\{w \in \mathbb{R} \mid a_w \neq 0\}$. The residue field of $\Delta((t^{\mathbb{R}}))$ is Δ, and the formal monomials t^w form the section $t^\bullet$.

The remainder of this subsection is devoted to prove a proposition that will be needed in subsection 4.2. We need to construct some small valued fields first. If k is a field, consider the field $k(t)$ of rational functions in one variable. There is only one real-valued valuation on this field that is zero on k^* and such that $v(t) = 1$ (see [7, Thm. 2.1.4]). The image of this valuation is $\mathbb{Z}$ and the residue field is k.

In a similar way it is possible to see that every valuation of $\mathbb{Q}$ is either zero on $\mathbb{Q}^*$, or it is the p-adic valuation, for a prime number p (see again [7, Thm. 2.1.4]). The image of this valuation is $\mathbb{Z}$ and the residue field is $\mathbb{F}_p$.

These fields are not algebraically closed, hence we need to extend the valuation to their algebraic closure. By [7, Thm. 3.1.1] every valuation of a field admits an extension to the algebraic closure, and by [7, Thm.3.4.3] this extension is again real-valued. There can be many different ways to construct this extension, but any two of them are related by an automorphism of the algebraic closure fixing the base field $k(t)$ or $\mathbb{Q}$, see [7, Thm. 3.2.15]. Hence, up to automorphisms, there exists a unique real-valued valuation of the field $\overline{k(t)}$ that is zero on k^* and such that $v(t) = 1$. This valuation has image $\mathbb{Q}$, residue field $\overline{k}$ and characteristic pair $(\mathrm{char}(k), \mathrm{char}(k))$. In the same way, up to automorphisms, there exists a unique valuation of $\overline{\mathbb{Q}}$ that restricts to the p-adic valuation on $\mathbb{Q}$. This valuation has image $\mathbb{Q}$, residue field $\overline{\mathbb{F}_p}$ and characteristic pair $(0,p)$.

We will denote by $\mathbb{F}_{(0,0)}$ the field $\overline{\mathbb{Q}(t)}$ with the valuation described above, by $\mathbb{F}_{(p,p)}$ the field $\overline{\mathbb{F}_p(t)}$ with the valuation described above, and by $\mathbb{F}_{(0,p)}$ the field $\overline{\mathbb{Q}}$ with the p-adic valuation. These valued fields are, in some sense, universal, as we show in the following proposition:

Proposition 3.1 Let $\mathbb{K}$ be an algebraically closed field, with a real-valued valuation $v : \mathbb{K}^* \mapsto \mathbb{R}$. Then $\mathbb{K}$ contains a subfield $\mathbb{F}$ such that $(\mathbb{F}, v|_{\mathbb{F}})$ is isomorphic as a valued field to one of the fields $\mathbb{F}_{(0,0)}, \mathbb{F}_{(p,p)}, \mathbb{F}_{(0,p)}$.

Proof. If $\mathrm{char}(\mathbb{K}) = p$, it contains a copy of $\mathbb{F}_p$. The valuation is zero on $\mathbb{F}_p^*$ because the multiplicative group $\mathbb{F}_p^*$ is cyclic. Choose an element t such that $v(t) = 1$. Then $\mathbb{K}$ contains a copy of $\overline{\mathbb{F}_p(t)}$. By the arguments above, the valuation restricted to this field is isomorphic to $\mathbb{F}_{(p,p)}$.

If char($\mathbb{K}$) = 0, it contains a copy of $\mathbb{Q}$. The valuation, restricted to $\mathbb{Q}^*$, can be zero or a p-adic valuation. If it is zero, choose an element t such that $v(t) = 1$. Then $\mathbb{K}$ contains a copy of $\overline{\mathbb{Q}(t)}$, and, by the arguments above, the valuation restricted to this field is isomorphic to $\mathbb{F}_{(0,0)}$. If it is the p-adic valuation, then $\mathbb{K}$ contains a copy of $\overline{\mathbb{Q}}$, and, by the arguments above, the valuation restricted to this field is isomorphic to $\mathbb{F}_{(0,p)}$. □

3.2. Tropical varieties

We denote by $\mathbb{T} = (\mathbb{R}, \oplus, \odot)$ the **tropical semifield**, with tropical operations $a \oplus b = \max(a, b)$ and $a \odot b = a + b$.

Let $\mathbb{K}$ be an algebraically closed field, with a real-valued valuation $v : \mathbb{K}^* \mapsto \mathbb{R}$, with image Λ_v. We often prefer to use the opposite function, and we will call it the **tropicalization**:

$$\begin{array}{rccc} \tau : & \mathbb{K}^* & \longmapsto & \mathbb{T} \\ & x & \longmapsto & -v(x) \end{array}$$

The **tropicalization map** on $(\mathbb{K}^*)^n$ is the component-wise tropicalization function:

$$\begin{array}{rccc} \tau : & (\mathbb{K}^*)^n & \longmapsto & \mathbb{T}^n \\ & (x_1, \ldots, x_n) & \longmapsto & (\tau(x_1), \ldots, \tau(x_n)) \end{array}$$

The image of this map is Λ_v^n, a dense subset of $\mathbb{R}^n$.

Let $I \subset \mathbb{K}[x_1, \ldots, x_n]$ be an ideal, and let $Z = Z(I) \subset \mathbb{K}^n$ be its zero locus, an affine algebraic variety. The image of $Z \cap (\mathbb{K}^*)^n$ under the tropicalization map will be denoted by $\tau(Z)$ and it is a closed subset of Λ_v^n. The **tropicalization** of Z is the closure of $\tau(Z)$ in $\mathbb{R}^n$, and it will be denoted by Trop(V). In particular, if v is surjective ($\Lambda_v = \mathbb{R}$) we have $\tau(V) = \text{Trop}(V)$. A **tropical variety** is the tropicalization of an algebraic variety. Note that in this way the notion of tropical variety depends on the choice of the valued field $\mathbb{K}$. We will show in subsection 4.2 that it actually depends only on the characteristic pair of $\mathbb{K}$ and on the value group Λ_v.

A **tropical polynomial** is a polynomial in the tropical semifield, an expression of the form:

$$\phi = \bigoplus_{a \in \mathbb{N}^n} \phi_a \odot x^{\odot a} = \max_{a \in \mathbb{N}^n} (\phi_a + \langle x, a \rangle)$$

where $\phi_a \in \mathbb{T} \cup \{-\infty\}$, there are only a finite number of indices $a \in \mathbb{N}^n$ such that $\phi_a \neq -\infty$. Note that here $x = (x_1, \ldots, x_n)$ is a vector of indeterminates. If all the coefficients ϕ_a are $-\infty$, then ϕ is the null polynomial. If ϕ is a non null tropical polynomial, the **tropical zero locus** of ϕ is the set

$$T(\phi) = \left\{ \omega \in \mathbb{T}^n \mid \text{ the max. in } \max_{a \in \mathbb{N}^n} (\phi_a + \langle x, a \rangle) \text{ is attained at least twice} \right\}$$

If ϕ is the null polynomial, then $T(\phi) = \mathbb{T}^n$.

If $f = \sum_{a \in \mathbb{N}^n} f_a x^a \in \mathbb{K}[x_1, \ldots, x_n]$ is a polynomial, the **tropicalization** of f is the tropical polynomial

$$\tau(f) = \bigoplus_{a \in \mathbb{N}^n} \tau(f_a) \odot x^{\odot a}$$

where $\tau(0) = -\infty$.

The fundamental theorem of tropical geometry (see [14, Thm. 3.2.4]) states that if $I \subset \mathbb{K}[x_1, \ldots, x_n]$ is an ideal, then

$$\mathrm{Trop}(Z(I)) = \bigcap_{f \in I} T(\tau(f))$$

It is also possible to find some finite systems of generators $f_1, \ldots, f_r$ of the ideal I such that

$$\tau(Z(I)) = T(\tau(f_1)) \cap \cdots \cap T(\tau(f_r))$$

Such a system of generators is called a **tropical basis**. It is possible to construct tropical bases with few polynomials of high degree, as in [12], but we also show that with the methods similar to the ones of [4, Thm. 2.9] it is possible to construct tropical bases made up of polynomials of controlled degree, see theorem 3.7.

The following proposition will be used later in subsection 4.2.

PROPOSITION 3.2 Let $(\mathbb{K}, v)$ be an algebraically closed valued field as above, let $\mathbb{F} \subset \mathbb{K}$ be an algebraically closed subfield such that $1 \in v(\mathbb{F})$, and consider the valued field $(\mathbb{F}, v|_{\mathbb{F}})$. Let $I \subset \mathbb{F}[x_1, \ldots, x_n]$ be an ideal, with zero locus $Z = Z(I) \subset \mathbb{F}^n$. Consider the extension $I^{\mathbb{K}}$ of I to $\mathbb{K}[x_1, \ldots, x_n]$, with zero locus $Z^{\mathbb{K}} \subset \mathbb{K}^n$. Then

$$\mathrm{Trop}(Z) = \mathrm{Trop}(Z^{\mathbb{K}})$$

Proof. One inclusion ($\mathrm{Trop}(Z) \subset \mathrm{Trop}(Z^{\mathbb{K}})$) is obvious because $Z \subset Z^{\mathbb{K}}$. To see the reverse inclusion, note that as $I \subset I^{\mathbb{K}}$, we have

$$\bigcap_{f \in I} T(\tau(f)) \supset \bigcap_{f \in I^{\mathbb{K}}} T(\tau(f))$$

and then use the fundamental theorem. □

If $I \subset \mathbb{K}[x_0, \ldots, x_n]$ is a homogeneous ideal, the variety $Z = Z(I)$ is a cone in $\mathbb{K}^{n+1}$: if $x \in Z$ and $\lambda \in \mathbb{K}$, then $\lambda x \in Z$. The image of Z in the projective space $\mathbb{KP}^n$ is a projective variety that we will denote by $V = V(I)$.

The tropicalization $\mathrm{Trop}(Z)$ is also a cone but in the tropical sense: if $\omega \in \mathrm{Trop}(Z)$ and $\lambda \in \mathbb{T}$, then $\lambda \odot \omega = (\lambda + \omega_1, \ldots, \lambda + \omega_n) \in \mathrm{Trop}(Z)$.

The **tropical projective space** is the set $\mathbb{TP}^n = \mathbb{T}^{n+1}/\sim$, where $\sim$ is the **tropical projective equivalence relation**:

$$x \sim y \Leftrightarrow \exists \lambda \in \mathbb{T} : \lambda \odot x = y \Leftrightarrow \exists \lambda \in \mathbb{R} : (x_0 + \lambda, \ldots, x_n + \lambda) = (y_0, \ldots, y_n)$$

We will denote by $[\cdot] : \mathbb{T}^{n+1} \mapsto \mathbb{TP}^n$ the projection onto the quotient, and we will use the homogeneous coordinates notation: $[x] = [x_0 : \cdots : x_n]$. To visualize $\mathbb{TP}^n$ it is possible to identify it with a subset of $\mathbb{T}^{n+1}$. If $i \in \{0, \ldots, n\}$, the subset $\{x \in \mathbb{T}^{n+1} \mid x_i = 0\}$ is the analog of an affine piece, and the quotient map $[\cdot]$, restricted to it, is a bijection. A more invariant way to do the same thing is to consider the set $\{x \in \mathbb{T}^{n+1} \mid x_0 + \cdots + x_n = 0\}$.

The image of $\mathrm{Trop}(Z)$ in the tropical projective space is a **tropical projective variety**. There is also a **projective tropicalization map**, defined on $\mathbb{KP}^n \setminus C$, where C is the union of the coordinate hyperplanes:

$$\tau : \mathbb{KP}^n \setminus C \ni [x] \mapsto [\tau(x_0) : \cdots : \tau(x_n)] \in \mathbb{TP}^n$$

The image of τ will be denoted by $\mathbb{P}(\Lambda_v^{n+1})$, a dense subset of $\mathbb{TP}^n$. If V is a projective variety in $\mathbb{KP}^n$, we will denote by $\tau(V)$ the image of $V \cap (\mathbb{KP}^n \setminus C)$ under the projective tropicalization map, a closed subset of $\mathbb{P}(\Lambda_v^{n+1})$, and by $\mathrm{Trop}(V)$ the closure of $\tau(V)$ in $\mathbb{TP}^n$.

3.3. Initial ideals

As before, let $\mathbb{K}$ be an algebraically closed field, with a real-valued valuation $v : \mathbb{K}^* \mapsto \mathbb{R}$, with image Λ_v.

Let $f = \sum_{a\in\mathbb{N}^n} f_a x^a \in \mathbb{K}[x_1, \ldots, x_n]$ be a non-zero polynomial, and let $\omega = (\omega_1, \ldots, \omega_n) \in \Lambda_v^n$ be a vector. Denote by H the number $\tau(f)(\omega)$. Now the polynomial $t^H f(t^{-\omega_1}x_1, \ldots, t^{-\omega_n}x_n) \in \mathcal{O}[x_1, \ldots, x_n] \setminus \mathfrak{m}[x_1, \ldots, x_n]$. Its image in $\Delta[x_1, \ldots, x_n]$ is a non-zero polynomial $\mathrm{in}_\omega(f)$, called the **initial form** of f in ω. If $f = 0$, we put $\mathrm{in}_\omega(f) = 0$.

Let $I \subset \mathbb{K}[x_1, \ldots, x_n]$ be an ideal. The initial ideal of I is the set $\mathrm{in}_\omega(I) = \{\mathrm{in}_\omega(f) \mid f \in I\} \subset \Delta[x_1, \ldots, x_n]$.

PROPOSITION 3.3 The set $\mathrm{in}_\omega(I)$ is an ideal of $\Delta[x_1, \ldots, x_n]$.

Proof. If $f \in \mathrm{in}_\omega(I)$, it is clear that for every monomial x^α, the product $x^\alpha f \in \mathrm{in}_\omega(I)$. We only need to verify that if $f, g \in \mathrm{in}_\omega(I)$ then $f + g \in \mathrm{in}_\omega(I)$. If $f + g = 0 = \mathrm{in}_\omega(0)$ there is nothing to prove, hence we can suppose that $f + g \neq 0$.

Let $F, G \in I$ such that $\mathrm{in}_\omega(F) = f$, $\mathrm{in}_\omega(G) = g$. Up to multiplying by an element of the section $t^\bullet$, we may suppose that $F(t^{\omega_1}x_1, \ldots, t^{\omega_n}x_n)$, $G(t^{\omega_1}x_1, \ldots, t^{\omega_n}x_n) \in \mathcal{O}[x_1, \ldots, x_n] \setminus \mathfrak{m}[x_1, \ldots, x_n]$. As $f + g \neq 0$, also $(F + G)(t^{\omega_1}x_1, \ldots, t^{\omega_n}x_n) \in \mathcal{O}[x_1, \ldots, x_n] \setminus \mathfrak{m}[x_1, \ldots, x_n]$. Hence $\mathrm{in}_\omega(F + G) = f + g$. □

Note that the definitions we have given of the initial form and initial ideal only work because we suppose that $\omega \in \Lambda_v^n$. If f is a polynomial, $\mathrm{in}_\omega(f)$ is a monomial if and only if the maximum in $\tau(f)(\omega)$ is attained only once. Therefore tropical varieties can be described in terms of initial ideals: $\tau(V(I))$ is the set of all $\omega \in \Lambda_v^n$ such that $\mathrm{in}_\omega(I)$ contains no monomials, and $\mathrm{Trop}(V(I))$ is the closure of this set.

LEMMA 3.4 Let $L \subset \mathbb{K}^n$ be a vector subspace. Denote by π the quotient of $\mathcal{O}$-modules $\pi : \mathcal{O}^n \longrightarrow \mathcal{O}^n/\mathfrak{m}^n = \Delta^n$. Then the image $L' = \pi(L \cap \mathcal{O}^n)$ is a vector subspace of Δ^n with $\dim_\Delta(L') = \dim_\mathbb{K}(L)$.

Proof. L' is an $\mathcal{O}$-submodule of Δ^n, hence it is a vector subspace. Put $h = \dim_\Delta(L')$. Let $v_1, \ldots, v_h$ be a basis of L', and complete it with vectors $v_{h+1}, \ldots, v_n$ to a basis of Δ^n. For every $i \leq h$ it is possible to find an element $w_i \in \mathcal{O}^n \cap L$ such that $\pi(w_i) = v_i$. For every $i > h$ it is possible to find an element $w_i \in \mathcal{O}^n$ such that $\pi(w_i) = v_i$, and w_i is necessarily not in L.

The elements $w_1, \dots, w_n$ are linearly independent over $\mathbb{K}$: if $a_1 w_1 + \cdots + a_n w_n = 0$, and some of the a_i are not zero, up to multiplying all the a_i's by an element of the section $t^\bullet$, we can suppose that all the a_i's are in $\mathcal{O}$, and some of them is not in $\mathfrak{m}$. Then $\pi(a_1)v_1 + \cdots + \pi(a_n)v_n = 0$ is a non-trivial linear combination.

Let $A = \mathrm{Span}(w_1, \dots, w_h)$ and $B = \mathrm{Span}(w_{h+1}, \dots, w_n)$. We know that $A \subset L$. We only need to show that $B \cap L = (0)$, this implies $L = A$.

Let $x \in B \cap L$, with $x = a_{h+1}w_{h+1} + \cdots + a_n w_n$. If $x \neq 0$, up to multiplying all the a_i's by an element of the section $t^\bullet$, we can suppose that all the a_i's are in $\mathcal{O}$, and some of them are not in $\mathfrak{m}$. Now $\pi(x) = \pi(a_{h+1})v_{h+1} + \cdots + \pi(a_n)v_n \in L'$ because $x \in L$. Hence $\pi(x) = 0$, and this forces all $\pi(a_{h+1}), \dots, \pi(a_n)$ to be zero, which is a contradiction. □

PROPOSITION 3.5 Let $I \subset \mathbb{K}[x_0, \dots, x_n]$ be a homogeneous ideal, and let $\omega \in \Lambda_v^{n+1}$. Then $\mathrm{in}_\omega(I)$ is a homogeneous ideal, with the same Hilbert function as I.

Proof. The map

$$\mathbb{K}[x_0, \dots, x_n] \ni f(x_0, \dots, x_n) \mapsto f(t^{\omega_1}x_1, \dots, t^{\omega_n}x_n) \in \mathbb{K}[x_0, \dots, x_n]$$

is an isomorphism of $\mathbb{K}[x_0, \dots, x_n]$. Up to this isomorphism, we can suppose that $\omega = 0$.

If $f \in \mathrm{in}_0(I)$, there is $F \in \mathcal{O}[x_1, \dots, x_n] \setminus \mathfrak{m}[x_1, \dots, x_n]$ such that $f = \mathrm{in}_0(F)$. The homogeneous components of f are then the initial forms of the homogeneous components of F not contained in $\mathfrak{m}[x_1, \dots, x_n]$. Hence $\mathrm{in}_0(I)$ is a homogeneous ideal.

The ideal $\mathrm{in}_0(I)$ is the image in $\Delta[x_0, \dots, x_n]$ of $I \cap \mathcal{O}[x_0, \dots, x_n]$, via the quotient map $\mathcal{O}[x_0, \dots, x_n] \mapsto \mathcal{O}[x_0, \dots, x_n]/\mathfrak{m}[x_0, \dots, x_n] = \Delta[x_0, \dots, x_n]$. For every $d \in \mathbb{N}$, the homogeneous component of degree d, $(\mathrm{in}_0(I))_d$, is the image in $(\Delta[x_0, \dots, x_n])_d$ of $I_d \cap (\mathcal{O}[x_0, \dots, x_n])_d$. By previous lemma, $\dim_\Delta((\mathrm{in}_0(I))_d) = \dim_{\mathbb{K}}(I_d)$. □

Note that even if I is a saturated homogeneous ideal, $\mathrm{in}_\omega(I)$ need not be saturated.

PROPOSITION 3.6 Let $I \subset \mathbb{K}[x_0, \dots, x_n]$ be a saturated homogeneous ideal, with Hilbert polynomial p. Then for every $\omega \in \Lambda_v^{n+1}$, if the initial ideal $\mathrm{in}_\omega(I)$ contains a monomial, it also contains a monomial of degree not greater than the Gotzmann number of p, here denoted by m_0.

Proof. If $\mathrm{in}_\omega(I)$ contains a monomial, then its saturation $(\mathrm{in}_\omega(I))^{\mathrm{sat}}$ also contains a monomial. By lemma 5.5 (to be proved in the section 5), $(\mathrm{in}_\omega(I))^{\mathrm{sat}}$ contains a monomial of degree m_0.

As $(\mathrm{in}_\omega(I))^{\mathrm{sat}}$ is saturated, its Hilbert function in degree m_0 is equal to its Hilbert polynomial evaluated in m_0, $p(m_0)$ (see the discussion in subsection 2.1). The Hilbert function of $\mathrm{in}_\omega(I)$ is equal to the Hilbert function of I. The ideal I is saturated, hence in degree m_0 also this Hilbert function is equal to the Hilbert polynomial $p(m_0)$. In particular, in degree m_0, the Hilbert functions of $\mathrm{in}_\omega(I)$ and its saturated are equal, hence their components in degree m_0 coincide. The monomial of degree m_0 we found in $(\mathrm{in}_\omega(I))^{\mathrm{sat}}$ is also in $\mathrm{in}_\omega(I)$. □

Now we use the previous proposition give a bound on the degree of a tropical basis of a saturated ideal in terms of its Hilbert polynomial. The tropical basis is constructed by adapting the methods of [4, Thm. 2.9] to the "non-constant coefficients" case.

THEOREM 3.7 Let $I \subset \mathbb{K}[x_0, \dots, x_n]$ be a saturated homogeneous ideal, with Hilbert polynomial p. As usual, we denote by m_0 the Gotzmann number of p. Then there exist a tropical basis $f_1, \dots, f_r \in I$ such that for all i, $\deg(f_i) \leq m_0$. In particular, if I_{m_0} denotes the component of degree m_0 of I, we have

$$\mathrm{Trop}(Z(I)) = \bigcap_{f \in I_{m_0}} T(\tau(f))$$

Proof. When ω varies in Λ_v^{n+1}, the initial ideals $\mathrm{in}_\omega(I)$ assume only a finite number of values (see [16, Thm. 2.2.1] or [14, Thm. 2.4.11]). We choose $\omega_1, \dots, \omega_p$ such that $\mathrm{in}_{\omega_1}(I), \dots, \mathrm{in}_{\omega_p}(I)$ are all the initial ideals containing monomials.

For every one of these ω_i we will construct a polynomial $f_i \in I$ in the following way. As $\mathrm{in}_{\omega_i}(I)$ contains a monomial, by the previous proposition we know that it also contains a monomial x^a of degree m_0. Choose a basis $g_1, \dots, g_h$ of $(\mathrm{in}_{\omega_i}(I))_{m_0}$ (the component of degree m_0) such that $g_1 = x^a$. As the set of all monomials of degree m_0 is a basis of $(\Delta[x_0, \dots, x_n])_{m_0}$, the independent set $g_1, \dots g_h$ may be extended to a basis of $(\Delta[x_0, \dots, x_n])_{m_0}$ by adding monomials $g_{h+1}, \dots g_N$. Now g_1 and $g_{h+1}, \dots g_N$ may also be interpreted as monomials in $(\mathbb{K}[x_0, \dots, x_n])_{m_0}$. As they are monomials, for every $\omega \in \Lambda_v^{n+1}$ we have $\mathrm{in}_\omega(g_j) = g_j$ (for $j = 1$ or $j > h$). By reasoning as in the proof of lemma 3.4, the elements $g_{h+1}, \dots, g_N$ give a basis of $(\mathbb{K}[x_0, \dots, x_n])_{m_0}/I_{m_0}$, hence there is a unique expression

$$g_1 = f_i + \sum_{j>h} c_j g_j$$

where $f_i \in I_{m_0}$. The polynomial f_i is the one we searched for. Note that its construction only depends on $\mathrm{in}_{\omega_i}(I)$, and not on the particular value of ω_i realizing this initial ideal. Now consider any ω such that $\mathrm{in}_\omega(I) = \mathrm{in}_{\omega_i}(I)$. We want to see that $\mathrm{in}_\omega(f_i) = g_1 = x^a$. This is because $\mathrm{in}_\omega(f_i)$ must be a linear combination of g_1 and the g_j with $j > h$. But the g_j with $j > h$ form a basis of a vector subspace of $(\Delta[x_0, \dots, x_n])_{m_0}$ that is complementary to $\mathrm{in}_\omega(I)$, while $\mathrm{in}_\omega(f_i)$ is in $\mathrm{in}_\omega(I)$. Hence $\mathrm{in}_\omega(f_i) = g_1$.

Now that we have constructed the polynomials $f_1, \dots, f_p$ we add to them other polynomials $f_{p+1}, \dots, f_r$ such that they generate the ideal I. We have to prove that, if $T = T(\tau(f_1)) \cap \dots \cap T(f_s)$, we have

$$\mathrm{Trop}(Z(I)) = T$$

The inclusion $\mathrm{Trop}(Z(I)) \subset T$ is clear. To show the reverse inclusion, first note that T is a finite intersection of tropical hypersurfaces, hence it is a Λ_v-rational polyhedral complex. In particular, $T \cap \Lambda_v^{n+1}$ is dense in T. For this reason we only need to verify that every $\omega \in T \cap \Lambda_v^{n+1}$ is in $\mathrm{Trop}(Z(I))$, or, conversely, that if $\omega \in \Lambda_v^{n+1}$ is not in $\mathrm{Trop}(Z(I))$, then it is not in T.

We have that $\mathrm{in}_\omega(I) = \mathrm{in}_{\omega_i}(I)$ for some i. Then the polynomial f_i has the property that $\mathrm{in}_\omega(f_i)$ is a monomial. Hence ω is not contained in $T(\tau(f_i))$. □

Note that the technique used in the previous proof to construct the polynomials f_i is similar to the division algorithm in standard Gröbner bases theory. The difference is that in this case the initial ideals $\mathrm{in}_\omega(I)$ are not, in general, monomial ideals hence the monomials not contained in $\mathrm{in}_\omega(I)$ do not form a basis for a complementary subspace. To overcome this problem we had to choose a subset of those monomials forming a basis for a complementary subset, hence the "remainder" of the division is not canonically determined, but it depends on this choice.

4. Applications

4.1. Tropicalization of the Hilbert Scheme

Let $\mathbb{K}$ be an algebraically closed field, with a real-valued valuation $v : \mathbb{K}^* \mapsto \mathbb{R}$. Denote by Λ_v the image of v, as above. Fix a projective space $\mathbb{KP}^n$, and consider a numerical polynomial $p(x) := \mathcal{P}(m_0, \ldots, m_s; x)$ such that $s < n$ and $m_0 \geq \cdots \geq m_s > 0$. To simplify notation, let $N := \binom{n+m_0}{n}$, and $M = \binom{N}{p(m_0)}$. Using the embedding described in subsection 2.3, we can identify the Hilbert scheme $\mathrm{Hilb}_n(p)$ with an algebraic subvariety of the projective space $\mathbb{KP}^{M-1}$, contained in the Plücker embedding of the Grassmannian $\mathbb{G}(N, N-p(m_0))$.

We construct the tropicalization of the Hilbert scheme using this embedding. As usual we denote by $\tau\left(\mathrm{Hilb}_n(p)\right) \subset \mathbb{P}\left(\Lambda^M\right)$ the image of the Hilbert scheme via the tropicalization map, and by $\mathrm{Trop}\left(\mathrm{Hilb}_n(p)\right)$ its closure in $\mathbb{TP}^{M-1}$. Of course, if $\Lambda_v = \mathbb{R}$, we have $\tau\left(\mathrm{Hilb}_n(p)\right) = \mathrm{Trop}\left(\mathrm{Hilb}_n(p)\right)$.

For every point $x \in \mathrm{Hilb}_n(p)$, denote by $V_x \subset \mathbb{KP}^n$ the algebraic subscheme parametrized by x.

Theorem 4.1 Let $x, y \in \mathrm{Hilb}_n(p) \subset \mathbb{KP}^{M-1}$. If $\tau(x) = \tau(y)$, then $\mathrm{Trop}(V_x) = \mathrm{Trop}(V_y)$.

Proof. Let I and J be the ideals corresponding, respectively, to V_x and V_y. The homogeneous parts I_{m_0}, J_{m_0}, considered as vector subspaces of S_{m_0}, corresponds to points of the Grassmannian $\mathbb{G}(N, N-p(m_0))$. By [17, Thm. 3.8] if two points of the Grassmannian have the same tropicalization, the linear spaces the parametrize have the same tropicalization. As we know that $\tau(x) = \tau(y)$, we have that $\mathrm{Trop}(I_{m_0}) = \mathrm{Trop}(J_{m_0})$. More explicitly, this means that

$$\{\tau(f) \mid f \in I_{m_0}\} = \{\tau(f) \mid f \in J_{m_0}\}$$

By theorem 3.7, this implies that $\mathrm{Trop}(V_x) = \mathrm{Trop}(V_y)$. □

We can sum up what we said so far in the following statement showing that the tropicalization of the Hilbert scheme can be interpreted as a parameter space for the set of tropical varieties that are the tropicalization of subschemes with a fixed Hilbert polynomial.

Theorem 4.2 There is a commutative diagram

$$\begin{array}{ccc} \mathrm{Hilb}_n(p) & \xleftrightarrow{\;b\;} & \{V \subset \mathbb{KP}^n \mid V \text{ has Hilbert polynomial } p\} \\ \big\downarrow \tau & & \big\downarrow \mathrm{Trop} \\ \tau(\mathrm{Hilb}_n(p)) & \xrightarrow{\;s\;} & \{\mathrm{Trop}(V) \subset \mathbb{TP}^n \mid V \text{ has Hilbert polynomial } p\} \end{array}$$

where b is the classical correspondence between points of the Hilbert scheme and subschemes of $\mathbb{K}\mathbb{P}^n$, and s is defined as

$$s(x) = \bigcap_{f \in L(x)} T(f)$$

where $L(x)$ is the tropical linear subspace of the space of tropical polynomials associated to the point x, seen as a point of the tropical Grassmannian. Moreover, the map s is surjective.

This parametrization is surjective, but it is in general not injective, as we will see in section 6. This result extends to the Hilbert schemes a property that was already known for the tropical Grassmannian (see [17, Thm. 3.8]).

4.2. Dependence on the valued field

The definition of tropical variety, as it was given in subsection 3.2, depends on the choice of a valued field $\mathbb{K}$. For an example of this, see [17, Thm. 7.2], where a tropical linear space is exhibited that is the tropicalization of a linear space over a field of characteristic 2 that cannot be obtained as tropicalization of a linear space over a field of characteristic 0.

Using the construction of transfinite Puiseux series, one can construct larger and larger valued fields, and it is *a priori* possible to think that if the field is sufficiently large, new tropical varieties may appear. As a corollary of our result about the Hilbert scheme, we show that that is not the case: the definition of tropical variety only depends on the characteristic pair of the valued field, and on the image group Λ_v. If we restrict our attention only to the largest possible image group, $\mathbb{R}$, we have that the definition of tropical variety only depends on the characteristic pair of the field.

Theorem 4.3 Let, $(\mathbb{K}, v), (\mathbb{K}', v')$ be two algebraically closed fields, with real-valued valuations $v : \mathbb{K}^* \mapsto \mathbb{R}$, $v' : \mathbb{K}'^* \mapsto \mathbb{R}$. Suppose that $\mathrm{cp}(\mathbb{K}) = \mathrm{cp}(\mathbb{K}')$, and $\Lambda_v \subset \Lambda_{v'}$. Let $V \subset \mathbb{K}\mathbb{P}^n$ be a subscheme. Then there exists a subscheme $W \subset \mathbb{K}'\mathbb{P}^n$ with the same Hilbert polynomial as V and such that $\mathrm{Trop}(V) = \mathrm{Trop}(W)$.

Proof. Let p be the Hilbert polynomial of V. Consider the Hilbert scheme $\mathrm{Hilb}_n(p)$. We claim that the tropical variety $\mathrm{Trop}(\mathrm{Hilb}_n(p))$ is the same for the two fields $\mathbb{K}$ and $\mathbb{K}'$. This is because, by proposition 3.1 both fields contain a subfield $\mathbb{F}$ isomorphic to one of the fields $\mathbb{F}_{(0,0)}, \mathbb{F}_{(p,p)}, \mathbb{F}_{(0,p)}$. As we said in subsection 2.3, the Hilbert scheme is defined by equations with integer coefficients, equations that are not dependent on the field. By proposition 3.2, $\mathrm{Trop}(\mathrm{Hilb}_n(p))$ is equal if defined over $\mathbb{F}$ or $\mathbb{K}$ or $\mathbb{K}'$.

Now if $x \in \mathrm{Hilb}_n(p)$ (over $\mathbb{K}$) is the point corresponding to V, we have that $\tau(x) \in \mathrm{Trop}(\mathrm{Hilb}_n(p)) \cap \mathbb{P}(\Lambda_v^M)$. As $\Lambda_v \subset \Lambda_{v'}$, there is a point $y \in \mathrm{Hilb}_n(p)$ (over $\mathbb{K}'$) such that $\tau(y) = \tau(x)$. Now let $W \subset \mathbb{K}'\mathbb{P}^n$ be the subscheme corresponding to y. By theorem 4.2, we have $\mathrm{Trop}(W) = \mathrm{Trop}(V)$. □

This theorem can be restated as follows:

Theorem 4.4 The set of tropical varieties definable over an algebraically closed valued field $(\mathbb{K}, v)$ only depends on the characteristic pair of $(\mathbb{K}, v)$, and on the image group Λ_v. If you suppose $\Lambda_v = \mathbb{R}$, the set of tropical varieties only depends on the characteristic pair of $(\mathbb{K}, v)$.

5. Existence of monomials of bounded degree

The aim of this section is to prove lemma 5.5, that was needed above.

5.1. Arithmetic Degree

Let $I \subset S$ be a homogeneous ideal. A **primary decomposition** of I is a decomposition

$$I = \bigcap_{i=1}^{k} Q_i$$

where the Q_i's are homogeneous primary ideals, no Q_i is contained in the intersection of the others, and the radicals of the Q_i's are distinct. The radicals $P_i = \sqrt{Q_i}$ are called prime ideals **associated** with I, and they do not depend on the choice of the decomposition (see [1, p. 51, Thm. 4.5, Thm. 7.13], [8, Sect. 3.5]).

For every $i \in \{1 \dots k\}$ let

$$I_i = \bigcap \{Q_j | P_j \subsetneq P_i\}$$

If P_i is a minimal prime, then $I_i = (1)$, while if P_i is embedded, then $I_i \subset P_i$. Note that, even if the primary ideals Q_j are not always uniquely determined by I, the ideals I_i and $I_i \cap Q_i$ are uniquely determined, see [1, Thm. 4.10]. The irrelevant ideal $(x_0, \dots, x_n)$ is associated with I if and only if I is not saturated. In this case, if $(x_0, \dots, x_n) = P_1$, then the saturation I^{sat} is equal to the ideal I_1.

Following [3, p. 27], for every $i \in \{1 \dots k\}$ we define the multiplicity of P_i in I (written $\text{mult}_I(P_i)$) as the length ℓ of a maximal chain of ideals

$$Q_i \cap I_i = J_\ell \subsetneq J_{\ell-1} \subsetneq \cdots \subsetneq J_1 = P_i \cap I_i$$

where every $J_j = R_j \cap I_i$ for some P_i-primary ideal R_j.

For every $r \in \{-1, \dots, n-1\}$, the **arithmetic degree** of I in dimension r is

$$\text{arith-deg}_r(I) = \sum_{\{i \ | \ \dim(P_i) = r\}} \text{mult}_I(P_i) \deg(P_i)$$

If $r > \dim(I) = \max \dim(P_i)$, then we have $\text{arith-deg}_r(I) = 0$. If $s = \dim(I)$, then we have $\text{arith-deg}_s(I) = \deg(I)$. For every $0 \leq r \leq \dim(I)$ we have $\text{arith-deg}_r(I) = \text{arith-deg}_r(I^{\text{sat}})$. The only prime ideal of dimension -1 is the irrelevant ideal, which has degree 1. The arithmetic degree in dimension -1 indicates how much I is non saturated: $\text{arith-deg}_{-1}(I) = \dim_k(I^{\text{sat}}/I)$.

The **arithmetic degree** of I is

$$\text{arith-deg}(I) = \sum_{i=0}^{n} \text{arith-deg}_r(I)$$

Note that the value $\text{arith-deg}_{-1}(I)$ does not appear in the arithmetic degree, in particular $\text{arith-deg}(I) = \text{arith-deg}(I^{\text{sat}})$.

THEOREM 5.1 Let I be a homogeneous ideal and let $g = \mathcal{P}(m_0, \ldots, m_s; x)$ be its Hilbert polynomial. Then for every $i \in \{0, \ldots, s\}$:

$$\sum_{r=i}^{s} \text{arith-deg}_r(I) \leq m_i$$

In particular

$$\text{arith-deg}(I) \leq m_0$$

Moreover these bounds are optimal: For every Hilbert polynomial g, there exists an ideal I with Hilbert polynomial g and such that for all $i \in \{0, \ldots, s\}$ we have $\sum_{r=i}^{s} \text{arith-deg}_r(I) = m_i$.

Proof. This fact can be deduced by putting together a few results from [11].

For every $r \geq 0$, $\text{arith-deg}_r(I) = n_r(X)$, where X is the closed subscheme of $\mathbb{P}^n$ defined by I and $n_r(X)$ is defined in [11, p. 21, Rem. 3]. See [15, p. 420] for details.

A tight fan is a particular kind of closed subscheme of $\mathbb{P}^n$, see the definition at the beginning of [11, Chap. 3]. If X is a tight fan with Hilbert polynomial g, then $\sum_{r=i}^{s} n_r(X) = m_i$ (see [11, Cor. 3.3]).

Given any closed subscheme X of $\mathbb{P}^n$ with Hilbert polynomial g, it is possible to construct a tight fan Y with Hilbert polynomial g and $n_r(X) \leq n_r(Y)$ (as in the proof of [11, Thm. 5.6]). Hence $\sum_{r=i}^{s} n_r(X) \leq \sum_{r=i}^{s} n_r(Y) = m_i$. □

5.2. Proof of lemma 5.5

We will use the following standard notation. Let $I \subset S$ be a homogeneous ideal and $f \in S$.

$$(I : f) = \{g \in S \mid gf \in I\}$$

$$(I : f^\infty) = \{g \in S \mid \exists n : gf^n \in I\}$$

See [1, Ex. 1.12] for some properties of $(I : f)$, for example $(I : fg) = ((I : f) : g)$ and $(\bigcap_i I_i : f) = \bigcap_i (I_i : f)$. This same properties also hold for $(I : f^\infty)$: $(I : fg^\infty) = ((I : f^\infty) : g^\infty)$ and $(\bigcap_i I_i : f^\infty) = \bigcap_i (I_i : f^\infty)$.

Let $I = \bigcap_{i=1}^{k} Q_i$ be a primary decomposition of I, as above, with $\sqrt{Q_i} = P_i$. Up to reordering, we can suppose that $f \in P_1, \ldots, P_h$ and $f \notin P_{h+1}, \ldots, P_k$.

LEMMA 5.2

$$(I : f^\infty) = \bigcap_{i=h+1}^{k} Q_i$$

Proof.

$$(I : f^\infty) = (\bigcap_{i=1}^{k} Q_i : f^\infty) = \bigcap_{i=1}^{k} (Q_i : f^\infty)$$

If $i \leq h$, we have $f \in P_i$, hence $f^m \in Q_i$ for some m, hence $(Q_i : f^m) = (1) = (Q_i : f^\infty)$.

If $i > h$, we have $f \notin P_i$, hence $(Q_j : f) = Q_j$ by [1, Lemma 4.4], and $(Q_j : f^\infty) = Q_j$. □

We will need a way for estimating multiplicities, and to do that we need to construct some strictly ascending chains.

Lemma 5.3 Consider ideals $J, Q \subset S$ and elements $f \in S$ and $l \in \mathbb{N}$. The following statements are equivalent:

1. There exists $a \in J$ such that $af^l \in Q$ and $af^{l-1} \notin Q$.
2. The chain
$$Q \cap J \subset (Q : f) \cap J \subset \cdots \subset (Q : f^l) \cap J$$
is strictly ascendant.
3. $(Q : f^{l-1}) \cap J \subsetneq (Q : f^l) \cap J$.

Proof. (1) ⇒ (2): For all $i \in \{1, \dots, l\}$, $af^{l-i} \in (Q : f^i) \cap J$, but $af^{l-i} \notin (Q : f^{i-1}) \cap J$.

(2) ⇒ (3): Trivial.

(3) ⇒ (1): Let a be any element of $(Q : f^l) \cap J \setminus (Q : f^{l-1}) \cap J$. Then $a \in J$, $af^l \in Q$, but $af^{l-1} \notin Q$. □

We will denote by $\ell_f(I)$ the minimum $l \in \mathbb{N}$ such that $(I : f^l) = (I : f^{l+1})$. Some l with this property always exist, because the ascending chain of ideals $I \subset (I : f) \subset (I : f^2) \subset \dots$ is stationary. We consider $f^0 = 1$, hence $\ell_f(I) = 0$ if and only if $I = (I : f)$. Note that if $l \geq \ell_f(I)$, then $(I : f^l) = (I : f^\infty)$.

Clearly, if J_i are ideals such that $I = \bigcap J_i$, then $\ell_f(I) \leq \sup \ell_f(J_i)$.

Lemma 5.4 Let I be an ideal with primary decomposition $I = \bigcap_{i=1}^k Q_i$. Then for every f:
$$\ell_f(I) \leq \sum_{f \in P_i} \mathrm{mult}_I(P_i)$$

Proof. We reorder the primes such that $f \in P_1, \dots, P_h$ and $f \notin P_{h+1}, \dots, P_k$. Let $L(I) = \sum_{i=1}^h \mathrm{mult}_I(P_i)$. We need to show that $\ell_f(I) \leq L(I)$.

We proceed by induction on k, the number of primary components. If $k = 1$ then $I = Q_1$ is primary. If $x \notin P_1$, then by [1, Lemma 4.4] $\ell_f(I) = L(I) = 0$. If $f \in P_1$, then the chain
$$Q_1 \subsetneq (Q_1 : f) \subsetneq (Q_1 : f^2) \subsetneq \cdots \subsetneq (Q_1 : f^{\ell_f(I)-1}) \subset P_1$$
is a strictly increasing chain of ideals, they are all P_1-primary by [1, Lemma 4.4], hence $\ell_f(I) \leq \mathrm{mult}_I(P_1) = L$.

For general k, we write $H_i = I_i \cap Q_i$, where I_i is as in subsection 5.1. As $I = \bigcap_{i=1}^k H_i$, then $\ell_f(I) \leq \max_{i=1}^k \ell_f(H_i)$. Moreover for all i, $L(H_i) \leq L(I)$. Hence we only need to prove our theorem for ideals of the form H_i: ideals having one primary component which is embedded in all the others.

We can suppose that $I = \bigcap_{i=1}^k Q_i$, with $P_i \subset P_1$ for all $i > 1$, with $f \in P_1$. Again we write $H_i = I_i \cap Q_i$. Let $s = \max_{i=2}^k \ell_f(H_i)$. Up to reordering we can suppose that $s = \ell_f(H_2)$, and that $f \in P_2$. By inductive hypothesis, as H_2 has less primary components than I, we know that $s \leq L(H_2)$. If $s = \ell_f(I)$ we are done. We can suppose that $r = \ell_f(I) - s > 0$. Now we will prove that $r \leq \mathrm{mult}_I(P_1)$, and this is enough because then we will have $\ell_f(I) \leq s + r \leq L(H_2) + \mathrm{mult}_I(P_1) \leq L(I)$.

Consider the ideal $(I : f^s) = (Q_1 : f^s) \cap \left(\bigcap_{i=2}^k (H_i : f^s)\right)$. For $i \geq 2$, $(H_i : f^s) = (H_i : f^\infty)$. In particular $J = \bigcap_{i=2}^k (H_i : f^s) = \bigcap_{i=h+1}^k Q_i$ by lemma 5.2. Now for $i \geq 0$ we have $(I : f^{s+i}) = (Q_1 : f^{s+i}) \cap J$. The following chain is strictly increasing:

$$(Q_1 : f^s) \cap J \subsetneq (Q_1 : f^{s+1}) \cap J \subsetneq \cdots \subsetneq (Q_1 : f^{s+r}) \cap J$$

By lemma 5.3 there is $a \in J$ such that $af^{s+r} \in Q$ and $af^{s+r-1} \notin Q$. As we said, for every $i \geq 2$ we have $J \subset (H_i : f^\infty) = (H_i : f^s)$. In particular, for every $i \geq 2$, $af^s \in H_i$. Hence $af^s \in I_1 = \bigcap_{i=2}^k H_i$. If $b = af^s \in I_1$, we have $bf^r \in Q$ and $bf^{r-1} \notin Q$. By lemma 5.3, we have the strictly ascending chain:

$$Q_1 \cap I_1 \subsetneq (Q_1 : f) \cap I_1 \subsetneq \cdots \subsetneq (Q_1 : f^r) \cap I_1$$

And this implies that $r \leq \mathrm{mult}_I(P_1)$, as required. □

LEMMA 5.5 Let I be an ideal with primary decomposition $I = \bigcap_{i=1}^k Q_i$, and let $B = \sum_{i=1}^k \mathrm{mult}_I(P_i)$. Then there exists a monomial x^α of degree not greater than B such that

$$(I : x^\alpha) = (I : x_0 \cdots x_n^\infty)$$

In particular, if I contains a monomial, then I contains a monomial of degree not greater than B.

If I is saturated, then $B \leq \text{arith-deg}(I)$. If I is saturated and with Hilbert polynomial $\mathcal{P}(m_0, \ldots, m_s; x)$, then $B \leq m_0$.

Proof. Up to reordering the primary components, we can suppose that there exist integers $1 = h_0 \leq \cdots \leq h_{n+1} \leq k$ such that for every $i \in \{0, \ldots, n\}$

$$x_i \in P_{h_i}, \ldots, P_{h_{i+1}-1} \text{ and } x_i \notin P_{h_{i+1}}, \ldots, P_k$$

Now let $\alpha_i = \sum_{i=h_i}^{h_{i+1}-1} \mathrm{mult}_I(P_i)$. By applying repeatedly lemmas 5.4 and 5.2 we get

$$(I : x^\alpha) = (\cdots (I : x_0^{\alpha_0}) : \cdots : x_n^{\alpha_n}) = (\cdots (I : x_0^\infty) : \cdots : x_n^\infty) = (I : x_0 \cdots x_n^\infty)$$

The ideal I contains a monomial if and only if $(I : x_0 \cdots x_n^\infty) = (1) = (I : x_0^{\alpha_0} \cdots x_n^{\alpha_n})$ and this happens if and only if I contains $x_0^{\alpha_0} \cdots x_n^{\alpha_n}$.

The last assertion follows from theorem 5.1. □

6. Examples

6.1. Hypersurfaces

Let $\mathbb{K}$ be an algebraically-closed field, with a surjective real-valued valuation $v : \mathbb{K}^* \mapsto \mathbb{R}$. Fix a projective space $\mathbb{KP}^n$ and a degree d. Consider the Hilbert polynomial $p(x) = \mathcal{P}(m_0, \ldots, m_{n-1}; x)$ with $m_0 = \cdots = m_{n-1} = d$. An ideal $I \subset \mathbb{K}[x_0, \ldots, x_n]$ has Hilbert polynomial p if and only if $I = (f)$ with f homogeneous with $\deg(f) = d$, hence $\mathrm{Hilb}_n(p)$ is the parameter space of hypersurfaces of $\mathbb{KP}^n$ of degree d. For such ideals I, the component I_d of degree d contains only the scalar multiples of f, hence it is a one-dimensional linear subspace of S_d, and all one-dimensional linear subspaces of S_d are of this form. The Grassmannian of one-dimensional subspaces of S_d is the projective space $\mathbb{P}(S_d) = \mathbb{KP}^{N-1}$ with $N := \binom{n+d-1}{n-1}$. This is the space of projective classes of polynomials of degree d. The embedding of the Hilbert scheme described in subsection 2.3 is just the identification of $\mathrm{Hilb}_n(p)$ with $\mathbb{KP}^{N-1}$. Its tropicalization is $\mathrm{Trop}\,(\mathrm{Hilb}_n(p)) = \mathbb{TP}^{N-1}$.

In this case it is possible to understand the correspondence from $\mathrm{Trop}\,(\mathrm{Hilb}_n(p))$ to the set of tropical hypersurfaces of degree d quite well. Here we want to underline two facts. One is that this map is not injective. The other is that it is possible to adjust things so that the map becomes injective. It is necessary to add some extra structure to the tropical hypersurfaces, namely to add weights to the maximal faces, as usual. Once this extra structure is considered, there exists a unique subpolyhedron $P \subset \mathrm{Trop}\,(\mathrm{Hilb}_n(p))$ such that the restriction of the correspondence to P is bijective. We think that this property of the existence of a subpolyhedron that is a "good" parameter space should probably be true also for the general Hilbert scheme, but in general it is not clear what is the suitable extra structure. In the example of the next subsection we show that in the general case the weights are not enough.

The coordinates in $\mathbb{KP}^{N-1}$ correspond to the coefficients of the polynomial, hence the tropicalization map $\tau : \mathbb{KP}^{N-1} \setminus C \mapsto \mathbb{TP}^{N-1}$ sends the projective class of a homogeneous polynomial f in the projective class of the tropical polynomial $\tau(f)$. Note that the tropicalization is defined only on $\mathbb{KP}^{N-1} \setminus C$, where C is the union of the coordinate hyperplanes. Hence we are dealing only with homogeneous polynomials of degree d containing all the monomials of degree d with a non-zero coefficient. This implies that all these polynomials have the same Newton polytope, a simplex that we will denote by $Q \subset \mathbb{R}^{n+1}$. Let $A = Q \cap \mathbb{Z}^{n+1}$ be the set of monomials of degree d. We use the theory of coherent triangulations (see [9, Chap. 7, Def. 1.3]) and coherent subdivisions (see [9, Chap. 7, Def. 2.3]). The secondary fan of (Q, A) (see [9, Chap. 7, C])) is a subdivision of the space $\mathbb{T}^N$ with polyhedral cones. Two points in the same projective equivalence class belong to the same cone, hence this subdivision passes to the quotient $\mathbb{TP}^{N-1}$. The maximal cones of the secondary fan correspond to coherent triangulations. The subpolyhedron P is the union of all the maximal cones corresponding to triangulations using all the points of A as vertices.

Consider a coherent triangulation not using all the points of A as vertexes, and consider a tropical polynomial f lying in the corresponding cone. The points of A not used in the triangulation correspond to monomials of f that never contribute to the maximum of the polynomial. If you perturb slightly the values of their coefficients, the tropical variety does not change, but the point of $\mathrm{Trop}\,(\mathrm{Hilb}_n(p))$ changes. This shows that the correspondence is not injective.

6.2. Pairs of points in the tropical projective plane

Let $\mathbb{K}$ be an algebraically closed field, with a surjective real-valued valuation $v : \mathbb{K}^* \mapsto \mathbb{R}$. Consider the Hilbert scheme $\mathrm{Hilb}_2(2)$ parameterizing subschemes of the projective plane $\mathbb{KP}^2$ with Hilbert polynomial $\mathcal{P}(2; x) = 2$. We study this example to show a more interesting situation where the correspondence from $\mathrm{Trop}\,(\mathrm{Hilb}_2(2))$ to the set of tropical varieties is not injective. A more complete analysis of this example has been subsequently given by Brodsky and Sturmfels in [5].

A subscheme of $\mathbb{KP}^2$ has Hilbert polynomial 2 if and only if it is a pair of distinct points, or a single point with a tangent space of dimension 1. The ideals of such schemes can be retrieved from their homogeneous component of degree 2, which is a 4-dimensional subspace of the vector space of homogeneous polynomials in 3 variables of degree 2, which has dimension 6. The Grassmannian of 4-dimensional subspaces of S_2 is embedded in the projective space $\mathbb{KP}^{14}$ and $\mathrm{Hilb}_2(2)$ is isomorphic to the symmetric product of two copies of $\mathbb{KP}^2$ blown up along the diagonal. The points outside the exceptional divisor correspond to the pairs of distinct points, while the points on the exceptional divisor correspond to singular schemes. We are interested in the latter.

As we said, the singular schemes we find in this setting have a unique point and a 1-dimensional tangent space at it. These schemes can be reconstructed from the data of the point and a line passing through it. In this example we are interested in considering schemes that have the same support while being different as schemes. We fix a point $p = [a : b : c] \in \mathbb{KP}^2$. To consider all the schemes supported at p one has to consider all the lines through p, which are given by polynomials of the type $f := lx_0 + mx_1 + nx_2$. These lines are parametrized by a $\mathbb{KP}^1$ and thus in $\mathrm{Hilb}_2(2)$ the locus of the points parameterizing all the schemes supported at p, Γ_p, contained in the exceptional divisor, is isomorphic to $\mathbb{KP}^1$. For simplicity, we will restrict ourselves to the generic case; in particular none of the coordinates of p or the coefficients of f is zero. We can reduce the number of variables involved: the line has to pass through the point, meaning that $n = -\frac{al+bm}{c}$, and since l,m and n are only defined up to a constant, we can choose $m = 1$. A scheme supported at p is then defined by the parameter l and we will denote it by Z_l.

Now we want to compute the point of the Hilbert scheme that corresponds to a scheme Z_l. This point is determined by the homogeneous component I_2 of the ideal I defining Z_l. To find a basis for I_2 we need four independent polynomials in it and, since I contains f, three of these can be x_0f, x_1f and x_2f. As our fourth generator we choose $(cx_0 - ax_2)^2$, which corresponds to one of the lines that we excluded counted twice. Since f cannot be $cx_0 - ax_2$ these polynomials are independent and form a basis for I_2. The wedge product of the four polynomials that form the basis is an element of $\bigwedge^4 S_2$. It is convenient to fix a basis and work with coordinates. As a basis of the vector space S_2 we choose the monomials in the order x_0^2, x_1^2, x_2^2, x_1x_2, x_0x_1, x_0x_1. As a basis of $\bigwedge^4 S_2$ we take the wedge products of four out of the six polynomials above, ordered in the same way, and we order the elements of this base in the lexicographic order.

Once we compute the wedge product and find the coordinates there are only four polynomials with more than one term appearing in these expressions, and they are $al + b$, $al - b$, $al + 2b$ and $2al + b$. To express the tropicalization of the coordinates we then need to divide 6 different cases: the first two where $v(al)$ and $v(b)$ are different and the other four where they are equal and the valuation of one of the four binomials is higher. For this reason, $\mathrm{Trop}(\Gamma_p)$ is the union of 6 rays coming out of a point P. Below we report the coordinates of P, in square brackets, and the six integer vectors defining each of the six rays. The letters A, B and C stand for the tropicalization of

a, b and c respectively.

$$\begin{bmatrix} A+B+C \\ 2B+C \\ B+2C \\ B+2C \\ 3C \\ -A+B+3C \\ 3B \\ 2B+C \\ -A+3B+C \\ -A+2B+2C \\ 2A+B \\ 2A+C \\ A+B+C \\ A+2C \\ A+2B \end{bmatrix}, \begin{pmatrix} 1\\0\\1\\1\\0\\1\\1\\1\\1\\1\\1\\0\\1\\0\\1 \end{pmatrix}, \begin{pmatrix} -1\\0\\0\\0\\0\\-1\\0\\0\\-1\\-1\\0\\0\\0\\0\\0 \end{pmatrix}, \begin{pmatrix} 0\\0\\-1\\0\\0\\0\\-1\\0\\0\\0\\-1\\0\\0\\0\\-1 \end{pmatrix}, \begin{pmatrix} 0\\0\\0\\-1\\0\\0\\0\\0\\0\\0\\0\\0\\0\\0\\0 \end{pmatrix}, \begin{pmatrix} 0\\0\\0\\0\\0\\0\\0\\-1\\0\\0\\0\\0\\0\\0\\0 \end{pmatrix}, \begin{pmatrix} 0\\0\\0\\0\\0\\0\\0\\0\\0\\0\\0\\0\\-1\\0\\0 \end{pmatrix}$$

Let us now analyze these six rays and understand which schemes' corresponding points lie on each ray (see Figure 1). The points on the first ray correspond to the schemes Z_l for which $L > B-A$. The tropicalization of the line tangent to Z_l is defined by the tropical polynomial $\tau(f)$ having coefficients L, 0 and $L+A-C$. The same is true for the points on the second ray except that they are those for which $L < B-A$ and the coefficients of $\tau(f)$ are L, 0 and $B-C$. The points on the third ray are those for which $v(al+b) < v(al) = v(b) = B$. For them the coefficients are $B-A$, 0 and $-v(al+b)$. Note that, together, these three expressions give us all the tropical lines passing through the point $[A : B : C]$, which is $\mathrm{Trop}(Z_l)$.

The points on the last three rays have a different behavior: for all of them and for the point P the tropicalization of the line is always the same and it is the one that has its center in $[A : B : C]$.

For any value of l we have that $\mathrm{Trop}(Z_l)$ is always the same point $[A : B : C]$. Even if one takes into account the weights (see [14, sec. 3.4]), the weight at $[A : B : C]$ is always 2. On the other hand there is a natural way to associate to any of the points

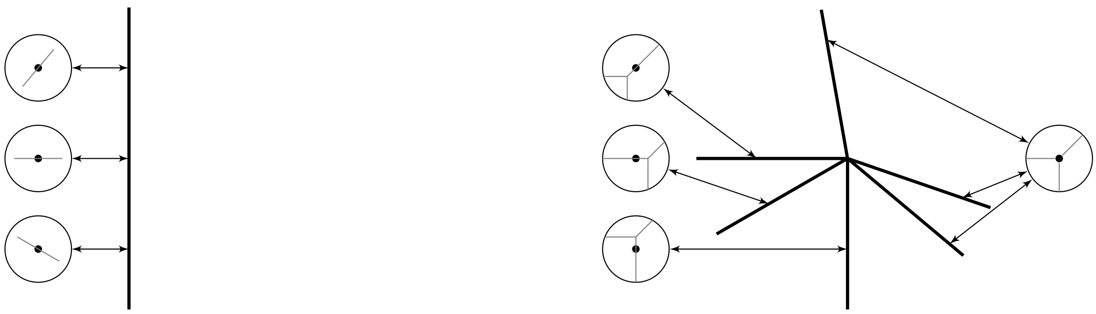

Figure 1: A point of Γ_p corresponds to a scheme in $\mathbb{P}^2$ supported at p with one tangent line. On the right we show what happens when we apply Trop to both p and the tangent line, depending on which ray of $\tau(\Gamma_p)$ the point lies on.

of $\mathrm{Trop}(\Gamma_p)$ a tropical line through $[A : B : C]$ that is the tropicalization of the tangent line to Z_l. This seems to suggest that one should consider some extra structure on $\mathrm{Trop}(Z_l)$ that includes the datum of the tropicalization of the tangent line.

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